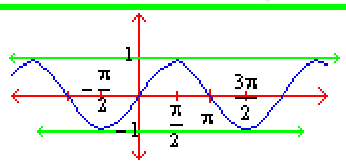
HOSSAM GHANEM

(8) 7.5 Inverse Trigonometric Functions (A)

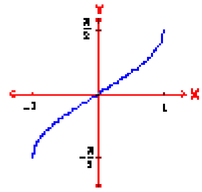


$y = \sin x$

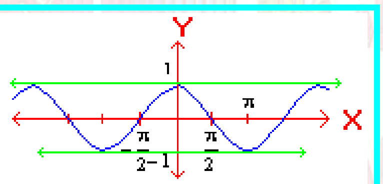
$$f(x) = \sin^{-1} x$$

$$y = \sin^{-1} x \Leftrightarrow \sin y = x$$

$$D_f = [-1\,,1\,] \qquad R_f = \left[-\frac{\pi}{2}\,,\frac{\pi}{2}\right]$$



$y = \sin^{-1} x$

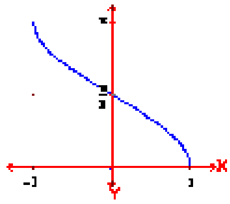


$y = \cos x$

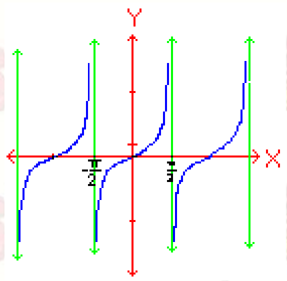
$$f(x) = \cos^{-1} x$$

$$y = \cos^{-1} x \Leftrightarrow \cos y = x$$

$$D_f = [-1\,,1\,] \qquad R_f = [0\,,\pi\,]$$



$y = \cos^{-1} x$

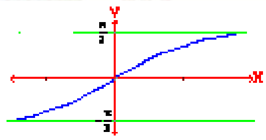


$y = \tan x$

$$f(x) = \tan^{-1} x$$

$$y = \tan^{-1} x \Leftrightarrow \tan y = x$$

$$D_f = \mathcal{R} \qquad R_f = \left(-\frac{\pi}{2}\,,\frac{\pi}{2}\right)$$

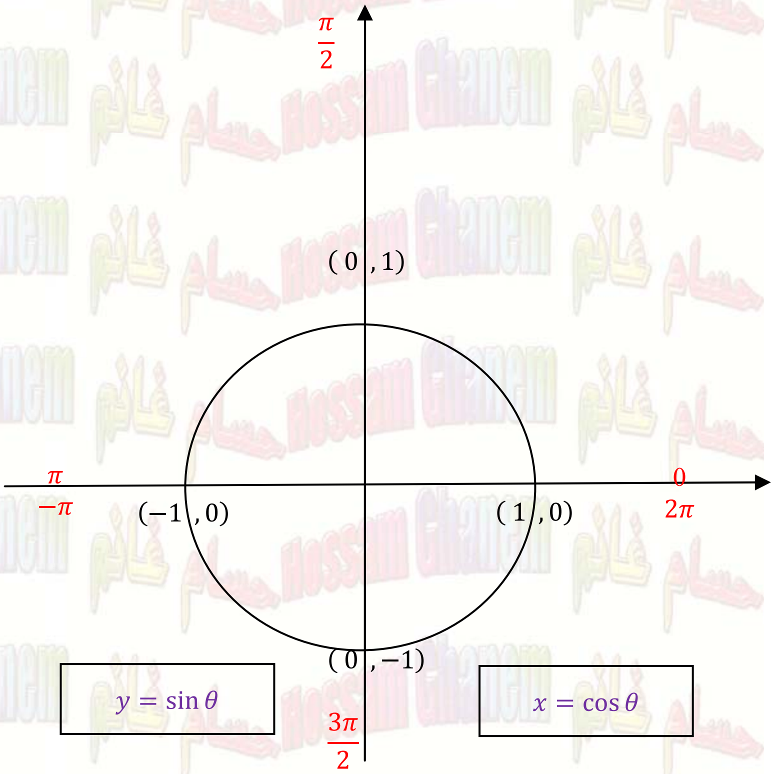


$y = \tan^{-1} x$

$f(x) = \sec^{-1} x$	$f(x) = \csc^{-1} x$	$f(x) = \cot^{-1} x$
$y = \sec^{-1} x \Leftrightarrow \sec y = x$	$y = \csc^{-1} x \Leftrightarrow \csc y = x$	$y = \cot^{-1} x \Leftrightarrow \cot y = x$
$D_f = \mathcal{R} \backslash (-1\,,1\,)$	$D_f = \mathcal{R} \backslash (-1\,,1\,)$	$D_f = \mathcal{R}$
$R_f = \left[0\,,\frac{\pi}{2}\right) \cup \left[\pi\,,\frac{3\pi}{2}\right)$	$R_f = \left[0\,,\frac{\pi}{2}\right) \cup \left[\pi\,,\frac{3\pi}{2}\right)$	$R_f = (0\,,\pi)$

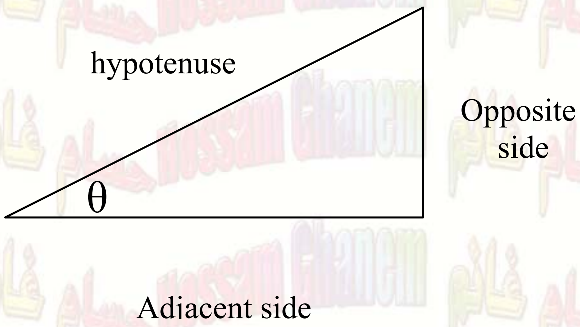
Corner Angles

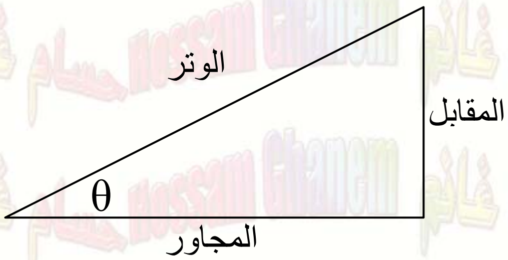
Angel		$\sin x$ y	$\cos x$ x	$\tan x$ y/x
2π	0	0	1	0
$-\frac{3\pi}{2}$	$\frac{\pi}{2}$	1	0	$\pm\infty$
$-\pi$	π	0	-1	0
$-\frac{\pi}{2}$	$\frac{3\pi}{2}$	-1	0	$\pm\infty$



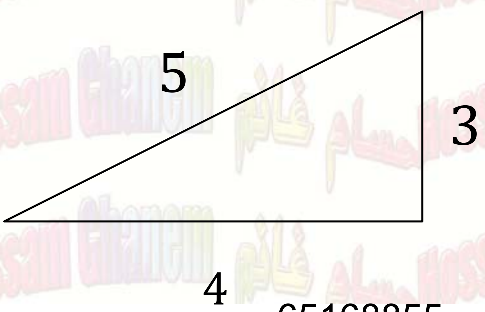
The Right Triangle

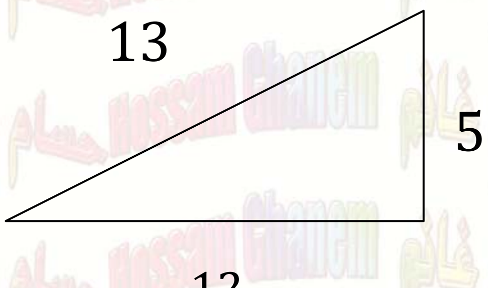
$\sin x$	المقابل / الوتر	opp / hyp
$\cos x$	المجاور / الوتر	adj / hyp
$\tan x$	المقابل / المجاور	opp / adj





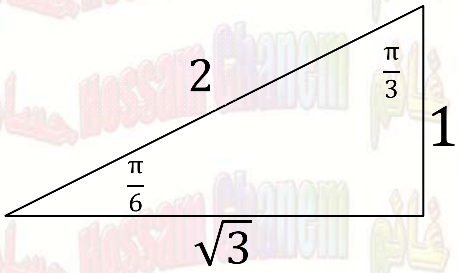
Special Triangles

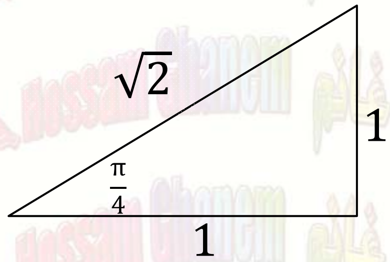




Special Angles

Angel		$\sin x$	$\cos x$	$\tan x$
$\frac{\pi}{6}$	30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$\frac{\pi}{3}$	60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$\frac{\pi}{4}$	45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1





Negative Angles

$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$	$\tan(-\theta) = -\tan\theta$

sin ↔ cos tan ↔ cot

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$	$\tan\left(\frac{\pi}{2} - \theta\right) = \cot\theta$
$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$	$\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$	$\tan\left(\frac{\pi}{2} + \theta\right) = -\cot\theta$

$\sin(\pi - \theta) = \sin\theta$	$\cos(\pi - \theta) = -\cos\theta$	$\tan(\pi - \theta) = -\tan\theta$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$	$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\tan(x + y) = \dfrac{\tan x + \tan y}{1 - \tan x \tan y}$	$\tan(x - y) = \dfrac{\tan x - \tan y}{1 + \tan x \tan y}$

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$\cos(2x) = \begin{cases} \cos^2 x - \sin^2 x \\ 2\cos^2 x - 1 \\ 1 - 2\sin^2 x \end{cases}$	$\sin(2x) = 2\sin x \cos x$

$\sin^{-1}(-x) = -\sin^{-1} x$	$\cos^{-1}(-x) = \pi - \cos^{-1} x$	$\tan^{-1}(-x) = -\tan^{-1} x$

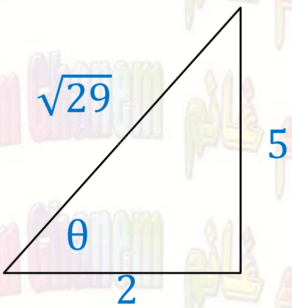
$\sin(\sin^{-1} x) = x$ for $-1 \le x \le 1$	$\cos(\cos^{-1} x) = x$ for $-1 \le x \le 1$
$\sin^{-1}(\sin x) = x$ for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$	$\cos^{-1}(\cos x) = x$ for $0 \le x \le \pi$

$\sin\theta = \dfrac{1}{\sqrt{1+\cot^2\theta}}$	$\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

Example 1
34 July 9, 2011

(2 pts) Find the exact value of $\sec\left(\tan^{-1}\left(\frac{5}{2}\right)\right)$

Solution



Let $\theta = \tan^{-1}\left(\frac{5}{2}\right)$

$\tan\theta = \frac{5}{2}$

$\sec\left(\tan^{-1}\left(\frac{5}{2}\right)\right) = \sec\theta = \frac{\sqrt{29}}{2}$

Example 2
29 July 2009 A

Find the value of $\cos^{-1}(\sin(5\pi/4))$

Solution

$$\cos^{-1}\left(\sin\left(\frac{5\pi}{4}\right)\right) = \cos^{-1}\left(\sin\left(\pi+\frac{\pi}{4}\right)\right) = \cos^{-1}\left(-\sin\frac{\pi}{4}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{2}}\right)$$

$$= \pi - \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

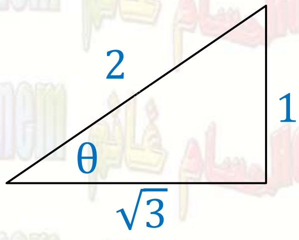
Example 3
19 March 2006 A

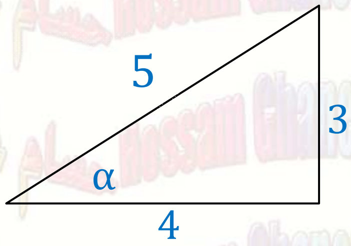
Find the exact value of $\cos\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)$

Solution

Let $\theta=\sin^{-1}\frac{1}{2}$ $\qquad$ $\alpha=\cos^{-1}\frac{4}{5}$

$\sin\theta=\frac{1}{2}$ $\qquad$ $\cos\alpha=\frac{4}{5}$





$$\cos\left(\sin^{-1}\left(\frac{1}{2}\right)+\cos^{-1}\left(\frac{4}{5}\right)\right)=\cos(\theta+\alpha)=\cos\theta\cos\alpha-\sin\theta\sin\alpha=\frac{\sqrt{3}}{2}\cdot\frac{4}{5}-\frac{1}{2}\cdot\frac{3}{5}=\frac{4\sqrt{3}-3}{10}$$

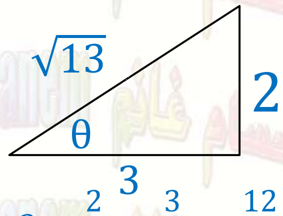
Example 4
14 March 2002

Calculate $\cos\left(2\tan^{-1}\left(\frac{2}{3}\right)-\left(\frac{\pi}{2}\right)\right)$

Solution

Let $\theta=\tan^{-1}\left(\frac{2}{3}\right)$

$\tan\theta=\frac{2}{3}$



$$\cos\left(2\tan^{-1}\left(\frac{2}{3}\right)-\left(\frac{\pi}{2}\right)\right)=\cos\left(2\theta-\frac{\pi}{2}\right)=\cos\left(\frac{\pi}{2}-2\theta\right)=\sin 2\theta=2\sin\theta\cos\theta=2\cdot\frac{2}{\sqrt{13}}\cdot\frac{3}{\sqrt{13}}=\frac{12}{13}$$

Example 5A
23 January 2005 A

statements is true or false-explain your answer $\cos^{-1}\left[\cos\left(-\frac{\pi}{4}\right)\right]=-\frac{\pi}{4}$

Solution

False

(1) $\cos^{-1}\left(\cos\left(\frac{-\pi}{4}\right)\right)=\cos^{-1}\left(\cos\left(\frac{\pi}{4}\right)\right)=\frac{\pi}{4}$

(2) $\cos^{-1}\left(\cos\left(\frac{-\pi}{4}\right)\right)\neq\frac{-\pi}{4}$, $\frac{-\pi}{4}\notin[0,\pi]$

Example 5B
33 April 10, 2011

Answer the following as true or False $\sin^{-1}(\sin 2)=2$

Solution

False $\sin^{-1}(\sin 2)\neq 2$, $2\notin\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

Example 6
6 March 1997

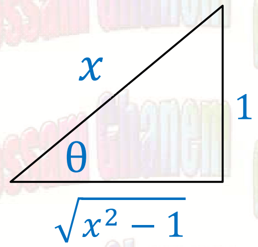
Write $\sec\left(\sin^{-1}\left(\dfrac{1}{x}\right)\right)$ as an algebraic expression in x if $x > 1$.

Solution

$\theta = \sin^{-1}\dfrac{1}{x}$

$\sin\theta = \dfrac{1}{x}$

$\therefore \sec\left(\sin^{-1}\dfrac{1}{x}\right) = \sec\theta = \dfrac{x}{\sqrt{x^2-1}}$



Example 7
14 January 2012

Prove the following (3 + 1 pts)

(a) $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ if $xy < 1$

(b) $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \pi/4$

Solution

(a)

Let $f(x) = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

$$f^{\backslash}(x) = \frac{1}{1+\left(\frac{x+y}{1-xy}\right)^2}\cdot\frac{(1-xy)(1)-(x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2+(x+y)^2}$$

$$= \frac{1+y^2}{1-2xy+x^2y^2+x^2+2xy+y^2} = \frac{(1+y^2)}{1+x^2y^2+x^2+y^2} = \frac{(1+y^2)}{(1+x^2)+(y^2+xy^2)}$$

$$= \frac{(1+y^2)}{(1+x^2)+y^2(1+x^2)} = \frac{(1+y^2)}{(1+x^2)(1+y^2)} = \frac{1}{(1+x^2)}$$

Let $g(x) = \tan^{-1}x$

$g^{\backslash}(x) = \dfrac{1}{(1+x^2)}$

$\therefore f^{\backslash}(x) = g^{\backslash}(x)$

$\therefore f(x) = g(x) + C$

$\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \tan^{-1}x + C$

$at\ \ x = 0$

$\tan^{-1}(y) = \tan^{-1}0 + C$

$C = \tan^{-1}y$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$

(b) $\tan^{-1}(1/2) + \tan^{-1}(1/3) = \tan^{-1}\left(\dfrac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right) = \tan^{-1}\left(\dfrac{\frac{3}{6}+\frac{2}{6}}{1-\frac{1}{6}}\right) = \tan^{-1}\left(\dfrac{\frac{5}{6}}{\frac{5}{6}}\right) = \tan^{-1}1 = \pi/4$

Example 8

32 Jan. 2009 A

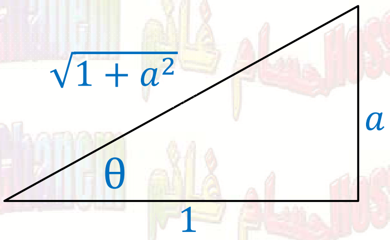
Show that if $\cos(\arctan a) = \tan(\arccos a)$, then $2a^2 = \sqrt{5} - 1$

Solution

$$\cos(\tan^{-1} a) = \tan(\cos^{-1} a)$$

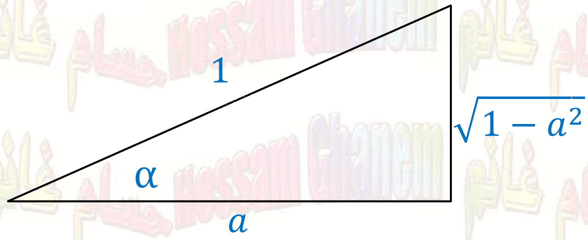
Let $\theta = \tan^{-1} a$

$\tan\theta = a$



Let $\alpha = \cos^{-1} a$

$\cos\alpha = a$



$$\cos(\tan^{-1} a) = \tan(\cos^{-1} a)$$
$$\cos\theta = \tan\alpha$$
$$\frac{1}{\sqrt{1+a^2}} = \frac{\sqrt{1-a^2}}{a}$$
$$a = \sqrt{1+a^2}\sqrt{1-a^2}$$
$$a = \sqrt{1-a^4}$$
$$a^2 = 1 - a^4$$
$$a^4 + a^2 - 1 = 0$$
$$a^2 = \frac{-1 \pm \sqrt{1+4}}{2}$$
$$2a^2 = \sqrt{5} - 1$$

Example 9

1 Jan. 1995

Solve the equation $\cos^{-1} x = \sec^{-1} x$

Solution

Let $\cos^{-1} x = \theta$ $\therefore \cos\theta = x$

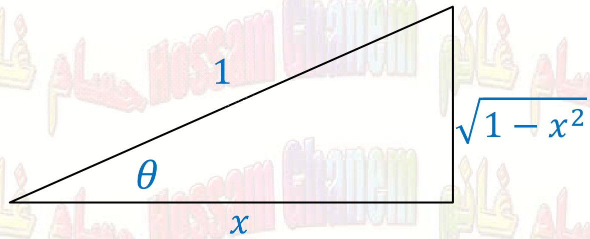
$\cos^{-1} x = \sec^{-1} x$

$\sec(\cos^{-1} x) = \sec(\sec^{-1} x)$

$\sec(\theta) = x$

$\frac{1}{x} = x$

$x^2 = 1$



$$x = \pm 1$$

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Homework

1	Write $\tan(\sin^{-1} 3x)$ as an algebraic expression in x	11 October 1999
2	Rewrite $\cos(\arctan(3x))$ as an algebraic expression of x if $x > 0$.	5 October 1996
3	Write $\sin(\tan^{-1} x) + \tan(\sin^{-1} x)$ as an algebraic expression in x for $x > 0$	8 October 1997
4	Find the exact value of $\sin\left(\cos^{-1}\left(\frac{2}{3}\right)\right)$	20 Nov. 2006 A
5	Evaluate $\sec\left(\sin^{-1}\frac{1}{3}\right) - \sin(\sec^{-1} 3)$	26 July 2008 A
6	Find the exact Value of $\cos\left(\tan^{-1}\left(\frac{1}{4}\right) - \frac{\pi}{3}\right)$.	10 March 1999
7	Calculate $\sin\left(\frac{\pi}{2} + \sec^{-1}(2)\right)$	21 March 2007 A
8	Find the exact value of $\cos\left(2\tan^{-1}\left(\frac{1}{3}\right) - \frac{\pi}{2}\right)$	28 April 2009 A
9	Determine whether the statement is true or false . Justify your answer for credit . (b) $\csc\left(\sin^{-1}\left(\frac{1}{x}\right)\right) = x$ for all $\lvert x \rvert \geq 1$.	(1 pt each)
10	Evaluate $\tan^{-1}\left(\tan\frac{7\pi}{6}\right) + \sin\left[\cos^{-1}\left(\frac{-1}{2}\right)\right]$	2 May 1995
11	Solve the following inequality for x $\tan^{-1}[2^x - 5] \geq 0$	32 Oct. 31st , 2010 A
12	Find the exact value of : $\cos\left(\cot^{-1}\left(\frac{4}{3}\right)\right)$	32 Oct. 31st , 2010 A
13	Find the exact value of $\sin\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$	27 June 2006 A

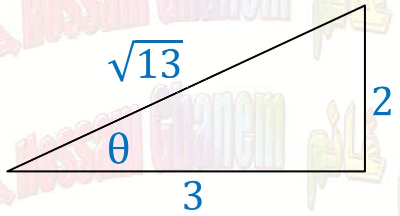
Homework

14	Write $\cos(\sec^{-1} x + \frac{\pi}{6})$ as an algebraic expression in x if $x \geq 1$. 2 March 1993

13
27 June 2006 A

Find the exact value of $\sin\left(\tan^{-1}\left(\frac{2}{3}\right)\right)$

Solution



$\theta = \tan^{-1}\left(\frac{2}{3}\right)$

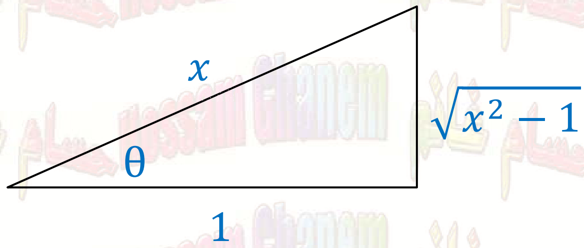
$\tan\theta = \frac{2}{3}$

$\therefore \sin\left(\tan^{-1}\left(\frac{2}{3}\right)\right) = \sin\theta = \frac{2}{\sqrt{13}}$

14
2 March 1993

Write $\cos(\sec^{-1} x + \frac{\pi}{6})$ as an algebraic expression in x if $x \geq 1$.

Solution



Let $\theta = \sec^{-1} x$

$\sec\theta = x$

$\cos(\sec^{-1} x + \frac{\pi}{6}) = \cos\left(\theta + \frac{\pi}{6}\right)$

$= \cos\theta\cos\frac{\pi}{6} - \sin\theta\sin\frac{\pi}{6}$

$= \frac{1}{x}\cdot\frac{\sqrt{3}}{2} - \frac{\sqrt{x^2-1}}{x}\cdot\frac{1}{2} = \frac{1}{2x}\left[\sqrt{3} - \sqrt{x^2-1}\right]$